

# Square Root of Metric: The Geometry Background of Unified Theory?

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We use the square root of inverse metric described by  $l = u^\dagger \gamma^a u \theta_a$  construct an  $U(4)$  gauge invariant, locally Lorentz invariant and generally covariant Lagrangian  $\mathcal{L} = \nabla l$  which describe  $U(4)$  non-abelian gauge theory in curved space-time. The particle's spectrum on this geometry is analyzed. We also construct a Lagrangian  $\mathcal{L}_g = \nabla^2 l^2$  which might describe gravity.

## INTRODUCTION

Einstein's geometry theory of gravity tells us that the gravitational field is originated from nontrivial metric of space-time. The Standard Model of Particle Physics [1–6] is a Non-Abelian gauge theory [7, 8] which has good correspondence with  $G$ -bundle [9]. If we can find a geometry structure equipped with nontrivial metric and  $G$ -bundle at the same time, we will find a geometry structure which might describe gravity and Non-Abelian gauge theory in a unified framework. But how to find this geometry structure naturally and give fermionic fields, gauge fields, gravitational field, scalar field intrinsic geometry interpretation? Inspired by Dirac's way of finding his equation and spinors through making square root of Klein-Gordon equation, we suggest to make the “square root” of the 4 dimensional Lorentzian manifold. This idea is similar with the ideas in articles [10–12]. But how to make “square root” for a manifold described by metric? After inverse metric be written as orthonormal frame, we just need to find the square root of  $\eta^{ab}$ . We know  $\eta^{ab}$  equals the trace of two gamma matrices. And we will prove the gamma matrices have  $U(4)$  freedom. Then, we find that the geometry object  $l = u^\dagger \gamma^a u \theta_a$  can be seen as square root of inverse metric. This  $U(4)$  freedom bring extra  $U(4)$ -bundle structure on 4 dimensional Lorentzian manifold such that “square root” of 4 dimensional Lorentzian manifold might be a better geometry description of real world space-time after we find isospin, color, flavor etc. Even though real world space-time might have deeper and more complex structure; but the geometrical framework of square root of inverse metric reasons enough to be taken seriously because it lead to a theory which is so close to the Standard Model, predictable and testable, satisfy all required symmetry, nothing added by hand and almost nothing leftover, having very simple formalism of Lagrangian and uniqueness mathematic structure.

The notations are introduced here.  $a, b, c, d$  represent frame indices, and  $a, b, c, d = 0, 1, 2, 3$ .  $\mu, \nu, \rho, \sigma$  represent coordinates indices, and  $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$ .  $\alpha, \beta, \gamma, \tau$  represent group indices, and  $\alpha = 0, 1, \dots, 15$ ,  $\beta = 1, 2, \dots, 8$ ,  $\gamma = 9, 11, 13$ ,  $\tau = 10, 12, 14$ . Repeated indices are summed by default.

## GEOMETRY AND LAGRANGIAN

The Pseudo-Riemannian manifold is described by metric

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu, g_{\mu\nu} = g_{\nu\mu}, \det(g_{\mu\nu}) \neq 0. \quad (1)$$

Here we discuss 4 dimensional Lorentzian manifold with signature  $(+, -, -, -)$ . The inverse metric is defined

$$g^{-1} = g^{\mu\nu} \partial_\mu \partial_\nu, \langle \partial_\nu, dx^\mu \rangle = \delta_\nu^\mu. \quad (2)$$

And it can be described by orthonormal frame formalism as

$$g^{-1} = \eta^{ab} \theta_a \theta_b. \quad (3)$$

Here  $\theta_a = \theta_a^\mu \partial_\mu$  are orthonormal frames and describe gravitational field,  $\eta^{ab} = \text{diag}(1, -1, -1, -1)$ .

The definition of gamma matrices is

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} I_{4 \times 4}. \quad (4)$$

The hermiticity conditions for gamma matrices can be chosen

$$\gamma^a \gamma^{b\dagger} + \gamma^{b\dagger} \gamma^a = 2I^{ab} I_{4 \times 4}, \quad (5)$$

here  $I^{ab} = \text{diag}(1, 1, 1, 1)$ . We define

$$\gamma'^a = \Lambda_b^a u^\dagger \gamma^b u. \quad (6)$$

If  $\gamma^b$  satisfy (4),  $u \in U(4)$  and  $\Lambda_b^a \in SO(1, 3)$ , direct calculation show that  $\gamma'^a$  satisfy (4) also. Here  $\Lambda_b^a$  can be absorbed into  $\theta_a$  but  $u$  not. So, we define

$$l = u^\dagger \gamma^a u \theta_a, \quad (7)$$

we find that

$$g^{-1} = \frac{1}{4} \text{tr}(l^2). \quad (8)$$

Then  $l$  is the square root of inverse metric  $g^{-1}$  in some sense.

Coefficients of affine connections on coordinates, coefficients of spin connections on orthonormal frame [13] and gauge fields on  $U(4)$ -bundle are defined as follows

$$\nabla_\mu \partial_\nu = \Gamma_{\nu\mu}^\rho \partial_\rho, \quad (9a)$$

$$\nabla_\mu \theta_a = \Gamma_{a\mu}^b \theta_b, \quad (9b) \quad \text{where}$$

$$\phi = \gamma^a \Gamma_{a\mu}^b \theta_b^\mu. \quad (12)$$

$$\nabla_\mu \gamma^a = \gamma^a A_\mu - A_\mu \gamma^a. \quad (9c)$$

$\Gamma_{a\mu}^b \theta_b^\rho = \partial_\mu \theta_a^\rho + \theta_a^\nu \Gamma_{\nu\mu}^\rho$  is found and  $A^{\mu\dagger} = -A^\mu$ . The uniqueness of definition of gauge fields is originated from restriction (4) and (5). Then the  $U(4)$  gauge invariant, locally Lorentz invariant and generally covariant Lagrangian is constructed as follows

$$\mathcal{L} = \nabla l. \quad (10)$$

This Lagrangian describe  $U(4)$  non-abelian gauge theory in curved space-time. The explicit formula of (10) is

$$\mathcal{L} = [(\partial_\mu u^\dagger - u^\dagger A_\mu) \gamma^a u + u^\dagger \gamma^a (\partial_\mu u + A_\mu u)] \theta_a^\mu + u^\dagger \phi u, \quad (11)$$

Compare (11) with Lagrangian [6] of the Standard Model of Particle Physics, take  $u$  as fermionic like fields here are reasonable. In flat space-time and using descartes coordinates,  $\theta_a^\mu = \delta_a^\mu$  and  $\phi = 0$  can be chosen. Then fermion propagators and boson-fermion interaction vertexes are determined by  $\partial_\mu u^\dagger \gamma^\mu u$ ,  $u^\dagger \gamma^\mu \partial_\mu u$ ,  $-u^\dagger A_\mu \gamma^\mu u$  and  $u^\dagger \gamma^\mu A_\mu u$ . Each term determine dozens of fermion propagators or boson-fermion interaction vertexes because  $u$  is  $4 \times 4$  matrix and  $A_\mu = iA_{\mu\alpha} \mathcal{T}_\alpha$ .

In curved space-time,  $\partial_\mu u^\dagger \gamma^a u \theta_a^\mu$ ,  $u^\dagger \gamma^a \partial_\mu u \theta_a^\mu$  describe fermionic like fields  $u$  propagating on curved space-time.  $-u^\dagger A_\mu \gamma^a u \theta_a^\mu$  and  $u^\dagger \gamma^a A_\mu u \theta_a^\mu$  describe boson-fermion interaction on curved space-time. Gravitational field is all other fields (except scalar field) dynamic background here which satisfy the characteristic of gravitational field in our real world. In Weyl basis of gamma matrices, the Lagrangian (10) can be preliminary decomposed as follows

$$\begin{aligned} \mathcal{L} = & \left( \partial_\mu u_R^\dagger \bar{\sigma}^a u_L + \partial_\mu u_L^\dagger \sigma^a u_R + u_R^\dagger \bar{\sigma}^a \partial_\mu u_L + u_L^\dagger \sigma^a \partial_\mu u_R \right) \theta_a^\mu + u_R^\dagger \bar{\sigma}^a u_L \Gamma_{a\mu}^b \theta_b^\mu + u_L^\dagger \sigma^a u_R \Gamma_{a\mu}^b \theta_b^\mu \\ & + iA_{\mu\beta} \left[ \left( u_R^\dagger \bar{\sigma}^a \quad u_L^\dagger \sigma^a \right) \mathcal{T}_\beta \begin{pmatrix} u_L \\ u_R \end{pmatrix} - \left( u_L^\dagger \quad u_R^\dagger \right) \mathcal{T}_\beta \begin{pmatrix} \sigma^a u_R \\ \bar{\sigma}^a u_L \end{pmatrix} \right] \theta_a^\mu \\ & + iA_{\mu\gamma} \left[ \left( u_R^\dagger \bar{\sigma}^a \quad u_L^\dagger \sigma^a \right) \mathcal{T}_\gamma \begin{pmatrix} u_L \\ u_R \end{pmatrix} - \left( u_L^\dagger \quad u_R^\dagger \right) \mathcal{T}_\gamma \begin{pmatrix} \sigma^a u_R \\ \bar{\sigma}^a u_L \end{pmatrix} \right] \theta_a^\mu \\ & + iA_{\mu\tau} \left[ \left( u_R^\dagger \bar{\sigma}^a \quad u_L^\dagger \sigma^a \right) \mathcal{T}_\tau \begin{pmatrix} u_L \\ u_R \end{pmatrix} - \left( u_L^\dagger \quad u_R^\dagger \right) \mathcal{T}_\tau \begin{pmatrix} \sigma^a u_R \\ \bar{\sigma}^a u_L \end{pmatrix} \right] \theta_a^\mu \\ & + iA_{\mu 15} \left[ \left( u_R^\dagger \bar{\sigma}^a \quad u_L^\dagger \sigma^a \right) \mathcal{T}_{15} \begin{pmatrix} u_L \\ u_R \end{pmatrix} - \left( u_L^\dagger \quad u_R^\dagger \right) \mathcal{T}_{15} \begin{pmatrix} \sigma^a u_R \\ \bar{\sigma}^a u_L \end{pmatrix} \right] \theta_a^\mu. \end{aligned}$$

Where  $u_L = \frac{1-\gamma^5}{2}u$ ,  $u_R = \frac{1+\gamma^5}{2}u$ ,  $\sigma^\mu = (1, \sigma^i)$ ,  $\bar{\sigma}^\mu = (1, -\sigma^i)$ . Hence, the chirality property of Lagrangian (10) is appeared.

Lagrangian (10) is demanded invariant under the transfer

$$u'^\dagger = u^\dagger U^\dagger, u' = Uu, \theta_a'^\mu = \Lambda_\nu^\mu \Lambda_a^b \theta_b^\nu, \gamma'^a = \Lambda_a^b U \gamma^b U^\dagger, \quad (13)$$

where  $U, U^\dagger \in U(4)$ ,  $\Lambda_a^b \in SO(1,3)$ ,  $\Lambda_\nu^\mu \in GL(4, \mathbb{R})$ . Then, the transformation rules have to be derived as follows

$$A'_\mu = \Lambda_\mu^\nu (U A_\nu U^\dagger - \partial_\nu U U^\dagger), \quad (14a)$$

$$\Gamma_{a\mu}^b = \Lambda_\mu^\nu (\Lambda_a^c \Gamma_{c\nu}^d \Lambda_d^b - \Lambda_a^c \partial_\nu \Lambda^b_c), \quad (14b)$$

$$\phi' = U \phi U^\dagger. \quad (14c)$$

Where  $\Lambda_\nu^\mu = \Lambda^\rho_\sigma g^{\mu\sigma} g_{\nu\rho}$  and  $\Lambda_a^b = \Lambda^d_c \eta^{bc} \eta_{ad}$  are used.

Curvature tensor and gauge field strength tensor are defined as follows

$$R_{b\mu\nu}^a = \partial_\mu \Gamma_{b\nu}^a - \partial_\nu \Gamma_{b\mu}^a + \Gamma_{b\nu}^c \Gamma_{c\mu}^a - \Gamma_{b\mu}^c \Gamma_{c\nu}^a, \quad (15a)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu. \quad (15b)$$

Here  $R_{ab\mu\nu} = -R_{ba\mu\nu}$  if  $\nabla g = 0$  and  $F_{\mu\nu}^\dagger = -F_{\mu\nu}$ . Define torsion  $T_{\nu\rho}^a = 2\Gamma_{[\nu\rho]}^a$ , we have Ricci identity and Bianchi identity [14] on this geometry structure as follows

$$R_{[\rho\mu\nu]}^a + \nabla_{[\rho} T_{\mu\nu]}^a = T_{\sigma[\rho}^a T_{\mu\nu]}^\sigma, \quad (16a)$$

$$\nabla_{[\rho} R_{b|\mu\nu]}^a = R_{b\sigma[\rho}^a T_{\mu\nu]}^\sigma, \quad (16b)$$

$$\partial_{[\mu} F_{\nu\rho]} = F_{[\mu\nu} A_{\rho]} - A_{[\mu} F_{\nu\rho]}. \quad (16c)$$

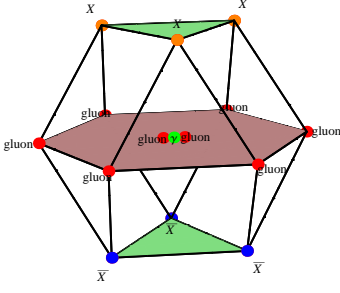


FIG. 1. Weight diagram of  $SU(4)$  adjoint representation and corresponding gauge bosons.

For gravity, Einstein-Hilbert action be showed as follows

$$S = \int R\omega. \quad (17)$$

Where  $R$  is Ricci scalar curvature,  $\omega = \sqrt{-g}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$  is volume form. And in this geometry framework, the equation can be derived as follows

$$\nabla_{[\mu} \nabla_{\nu]} l = \frac{1}{2}(u^\dagger \gamma^a F_{\mu\nu} u - u^\dagger F_{\mu\nu} \gamma^a u) \theta_a + \frac{1}{2} u^\dagger \gamma^a u R^b_{a\mu\nu} \theta_b. \quad (18)$$

Define  $\nabla^2 = \nabla_{[\mu} \nabla_{\nu]} dx^\mu \wedge dx^\nu$ , the Lagrangian  $\mathcal{L}_g$  might describe gravity is constructed

$$\mathcal{L}_g = \nabla^2 l^2. \quad (19)$$

This Lagrangian  $\mathcal{L}_g$  is obviously  $U(4)$  gauge invariant, locally Lorentz invariant and generally covariant. The explicit formula of Lagrangian  $\mathcal{L}_g$  is

$$\mathcal{L}_g = u^\dagger \gamma^a F_{ba} \gamma^b u + \frac{1}{2} u^\dagger F_{ab} \gamma^a \gamma^b u + \frac{1}{2} u^\dagger \gamma^a \gamma^b F_{ab} u - RI_{4 \times 4}. \quad (20)$$

Here  $\partial_\mu dx^\nu \otimes dx^\rho \partial_\sigma = \delta_\mu^\nu \delta_\sigma^\rho$ ,  $dx^\mu \otimes dx^\nu \partial_\rho \partial_\sigma = \delta_\rho^\mu \delta_\sigma^\nu$  are used and  $F_{ab} = F_{\mu\nu} \theta_a^\mu \theta_b^\nu$ . The  $RI_{4 \times 4}$  correspond to Einstein-Hilbert action naturally, the leftover parts might describe Lagrangian of matter in gravity theory. The action might describe gravity is  $S = \int \mathcal{L}_g \omega$ .

## PARTICLES SPECTRUM

$A_\mu$  are  $U(4)$  gauge fields and can be expanded by generators of  $U(4)$  be showed on appendix.

$$A_\mu = iA_{\mu\alpha} \mathcal{T}_\alpha. \quad (21)$$

Only traceless  $SU(4)$  generators being discussed below then  $\alpha = 1, 2, \dots, 15$ . Gauge field transfer as adjoint representation according to (13).  $SU(4)$  being decomposed as follows according to the weight diagram [15] Fig. 1.

$$SU(4) = SU(3) \oplus U(1) + U_X + U_{\bar{X}}. \quad (22)$$

TABLE I. Gauge bosons and corresponding  $SU(4)$  generators.

$SU(4)$			
$SU(3)$	$U_X$	$U_{\bar{X}}$	$U(1)$
<b>8</b>	<b>3</b>	<b><math>\bar{3}</math></b>	<b>1</b>
gluon	$X$	$\bar{X}$	$\gamma$
$\mathcal{T}_\beta$	$\mathcal{T}_\gamma$	$\mathcal{T}_\tau$	$\mathcal{T}_{15}$

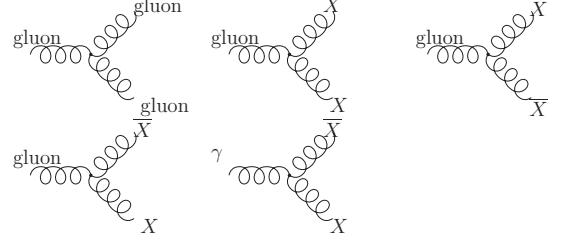


FIG. 2. 3-boson vertexes on flat spacetime.

The correspondence between  $SU(4)$  generators and gauge bosons is showed in Table I.

Electrodynamics like and Chromodynamics like theory appear in  $U(1)$  part and  $SU(3)$  part of this geometry framework in flat space-time naturally. For gauge bosons of weak interaction,  $W^\pm$  and  $Z$  are color-singlet is required; the only material leftover to construct  $W^\pm$  and  $Z$  are  $A_{\mu\gamma} \mathcal{T}_\gamma$  and  $A_{\mu\tau} \mathcal{T}_\tau$ ;  $F_{\mu\nu} = 0$  is most simple and reasonable setting here for free  $Z$  boson, particle and antiparticle need to be seen for  $W^\pm$  bosons. Under those 3 restrictions, simplest representation for  $W^\pm$  and  $Z$  are  $(\mathbf{3} \pm \mathbf{\bar{3}})^3$  and  $\mathbf{3} \otimes \mathbf{\bar{3}}$  and the explicit formalism of  $W^\pm$  and  $Z$  are exhibited as follows

$$W_\mu^\pm = \frac{1}{3!} \epsilon_\mu^{\nu\rho\sigma} A_{\nu 9} A_{\rho 11} A_{\sigma 13} \mathcal{T}_{[9}^\pm \otimes \mathcal{T}_{11}^\pm \otimes \mathcal{T}_{13}^\pm], \quad (23a)$$

$$\partial_{[\mu} Z_{\nu]} = A_{\mu\gamma} A_{\nu(\gamma+1)} \mathcal{T}_{[\gamma} \otimes \mathcal{T}_{\gamma+1]}. \quad (23b)$$

Where  $\mathcal{T}_\gamma^\pm = \mathcal{T}_\gamma \pm i\mathcal{T}_{\gamma+1}$  and  $A_{\mu 9} A_{\nu 10} = A_{\mu 11} A_{\nu 12} = A_{\mu 13} A_{\nu 14}$  in (23b). Here  $W^\pm$  and  $Z$  are composite particles and have nonzero mass. So the weak bosons appearing naturally on this geometry framework in flat space-time also. Now, this geometry structure in flat space-time lead to a  $SU(4)$  gauge theory might give unified framework for Chromodynamics and Electroweak theory in Standard Model of Particle Physics.

In flat space-time, the 3-boson vertexes and 4-boson vertexes determined by  $SU(4)$  structure constants. The 3-boson vertexes of this geometry framework in flat space-time are showed on Fig. 2.

Fermionic like fields  $u$  transfer as  $U(4)$  fundamental representation according to (13). So, fermions are filled into  $SU(4)$  fundamental representation naturally as Table II. In Table II, there exist both left handed and

TABLE II. Fermions be filled into  $SU(4)$  fundamental representation  $\mathbf{4} \otimes \mathbf{6}$ .

$SU(4)$		$\mathbf{6}$						
$\mathbf{4}$	Quarks	R						
		G	u	c	t	d	s	b
	Leptons	B						
			$e$	$\mu$	$\tau$	$\nu_e$	$\nu_\mu$	$\nu_\tau$

TABLE III. Corresponding relations between Quarks quantum number and weight diagram coordinates in Chevalley basis of  $SU(4)$  representation  $\mathbf{6}$ .

	$2I_z$	S+C	B+T	$H_1$	$H_2$	$H_3$
u	1	0	0	1	-1	1
d	-1	0	0	-1	1	-1
c	0	1	0	1	0	-1
s	0	-1	0	-1	0	1
t	0	0	1	0	1	0
b	0	0	-1	0	-1	0

right handed fermions for all quarks and leptons. Especially, the existence of right handed neutrinos is predicted. This is compatible with experimental results [16] and the well know Seesaw mechanism. The representation  $\mathbf{4}$  lead us reobtain “Lepton number as the fourth color” [17]. The weight diagram coordinates in Chevalley basis of representation  $\mathbf{6}$  have good correspondence with quarks quantum number [18] and we show it in Table III. Antifermions be filled into  $\mathbf{6} \otimes \bar{\mathbf{4}}$  similarly.  $2I_z$  being used in Table III are not no reasons as we have Gell-Mann-Nishijima formula [18]

$$Q = I_z + \frac{B + S + C + B + T}{2}.$$

$\phi$  is scalar field based on (12) and (14c). Corresponding particle of scalar field is Higgs. (12) tell us that scalar field origin from gravitational field. This is amazing but inescapable result in this geometry framework. In Standard Model, Yukawa coupling and gravity interaction without repulsion forces either but gauge interaction not at least on tree level. This might hint that there are deep connections between scalar field and gravitational field.

## CONCLUSION

The “square root” of 4 dimensional Lorentzian manifold have extra  $U(4)$ -bundle structure than 4 dimensional Lorentzian manifold. Fermionic like fields are originated from sections of  $U(4)$ -bundle, gauge fields are connections of  $U(4)$ -bundle, gravitational field is described by orthonormal frame, scalar field is originated from gravitational field. Fermionic like fields transfer as fundamental

representation and give quarks and leptons particles spectrum, gauge fields transfer as adjoint representation and give 8 gluons, color-singlet composite particles  $W^\pm$  and  $Z$ , 1 most familiar particle  $\gamma$ . The interactions between those fields are  $U(4)$  non-abelian gauge theory in curved space-time and be described by Lagrangian  $\mathcal{L} = \nabla l$ . Gravity might be described by Lagrangian  $\mathcal{L}_g = \nabla^2 l^2$ . If “square root” of 4 dimensional Lorentzian manifold is the better description of real world space-time,  $W^\pm$  and  $Z$  must be composite particles and right handed neutrinos should be exist.

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## Generators of $U(4)$

$$\begin{aligned}
\mathcal{T}_1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_8 = \frac{\sqrt{3}}{6} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_9 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_{10} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_{11} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathcal{T}_{12} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
\mathcal{T}_{13} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathcal{T}_{14} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},
\end{aligned}$$

$$\mathcal{T}_{15} = \frac{\sqrt{6}}{12} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \mathcal{T}_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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